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7 Pulse Modulation

In Section 6.1 we saw that bandlimited continuous-time signals can be represented by a sequence of discrete-time samples. Moreover, in Section 6.3, we saw that the continuous-time signal can be reconstructed if the sampling rate is sufficiently high.

Because the sequence $m[n]$ completely contains the information about $m(t)$, instead of trying to send $m(t)$, we may consider transmitting our message via $m[n]$ in the form of pulse modulation.

7.1 Analog Pulse Modulation

7.1. In this section, we start with a sequence of numbers (discrete-time message):

$$\cdots, m[-3], m[-2], m[-1], m[0], m[1], m[2], m[3], \cdots$$

as shown in Figure 55.

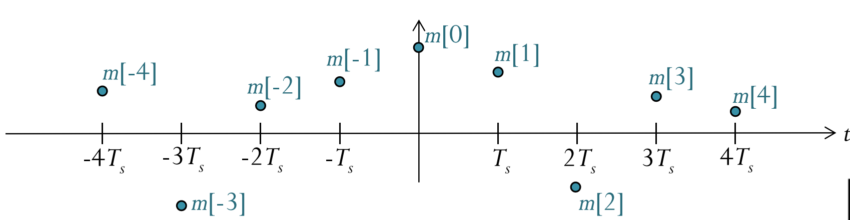


Figure 55: Sequence of Numbers for PAM

- These numbers may come from the process of sampling a continuous-time signal $m(t)$.
- Alternatively, it may directly represent (digital) information that intrinsically available in discrete-time. See Example 7.2.
- Because the $m[n]$ may not come from sampling, we call each $m[n]$ a **symbol**.

Example 7.2. Naturally digital information is an ordered sequence of symbols (or characters). Each symbol is drawn from an **alphabet** of $M \geq 2$ different symbols.

- English text: 26 (a to z) + 26 (A to Z) + 10 (0 to 9) + Punctuation and Other Signs (. , ! @ ( ))
  - Text is commonly encoded using ASCII, and MATLAB automatically represents any string file as a list of ASCII numbers.
- Thai text: 44 (Consonants) + 15 (Vowel Symbols) + 4 (Tone Marks) $+\ldots$
- A typical computer terminal has an alphabet of $M \approx 90$ symbols (the number of character keys multiplied by two to account for the shift key)

Definition 7.3. In **analog pulse modulation**, some attribute of a pulse varies continuously in one-to-one correspondence with a sample value.

- Example of a pulse:

- Three attributes can be readily varied: amplitude, width, and position.
- These lead to pulse-amplitude modulation (PAM), pulse-width modulation (PWM), and pulse-position modulation (PPM) as illustrated in Figure 57.

Definition 7.4. Unmodulated pulse train: $\sum_{n=-\infty}^{\infty} p(t - nT_s)$

- The pulse is repeated every T_s seconds.
- This replaces the role of a sinusoidal carrier.

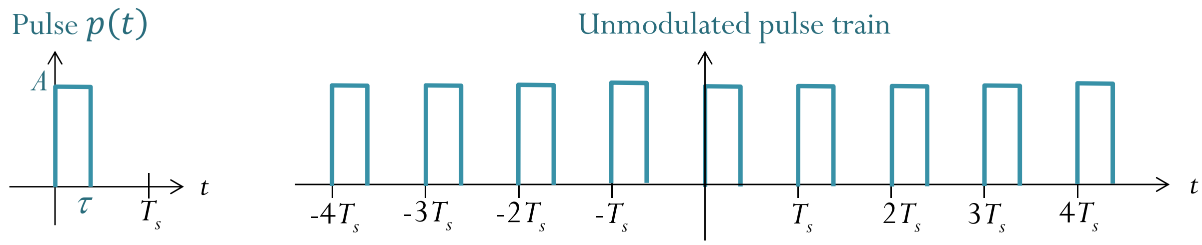


Figure 56: Unmodulated pulse train

Example 7.5. Figure 57 compares three types of analog pulse modulation.

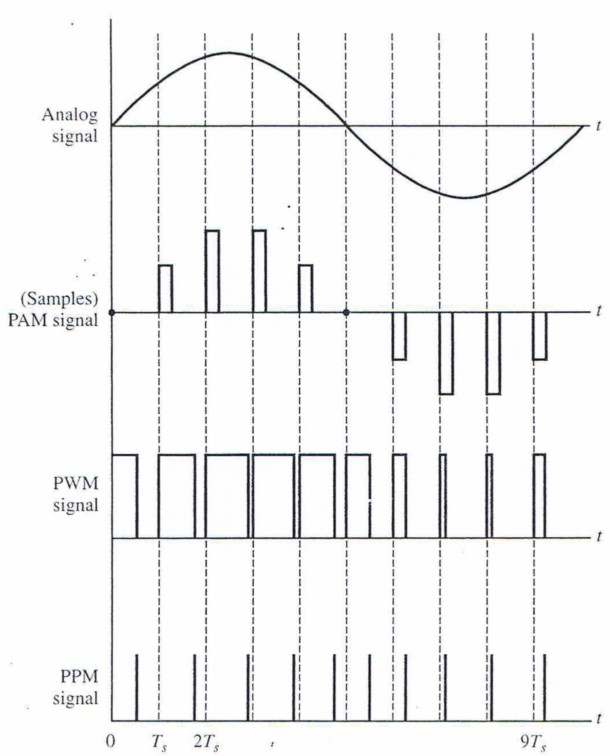


Figure 57: Illustration of PAM, PWM, and PPM. [14, Fig. 3.56]

Definition 7.6. In ***Pulse-Amplitude Modulation*** (PAM), the sample values modulate the amplitude (height) of a pulse train. We will focus on this type of modulation in Section 7.2.

Definition 7.7. ***Pulse-Width Modulation*** (PWM): A PWM waveform consists of a sequence of pulses with the width of the nth pulse is proportional to the value of $m[n]$.

- Seldom used in modern communications systems.
- Used extensively for DC motor control in which motor speed is proportional to the width of the pulses . Since the pulses have equal amplitude, the energy in a given pulse is proportional to the pulse width.

Definition 7.8. ***Pulse-Position Modulation*** (PPM): A PPM signal consists of a sequence of pulses in which the pulse displacement from a specified time reference is proportional to the sample values of the information-bearing signal.

- Have a number of applications in the area of ultra-wideband communications.

7.9. Pulse-modulation scheme are really baseband coding schemes, and they yield baseband signal.

7.2 Pulse-Amplitude Modulation

Definition 7.10. In ***Pulse-Amplitude Modulation*** (PAM), the sample values modulate the amplitude (height) of a pulse train. The pulse-modulated signal has the form

$$x_{\mathrm{PAM}}(t) = \sum_{n=-\infty}^{\infty} m[n]\, p(t - nT_s) \tag{85}$$

Example 7.11. An example of a PAM signal is shown in Figure 58.

Example 7.12. Another example of a PAM signal is shown in Figure 59. Note that the discrete-time sequence $m[n]$ is the same as in Example 7.11. However, the pulses used in these examples are different.

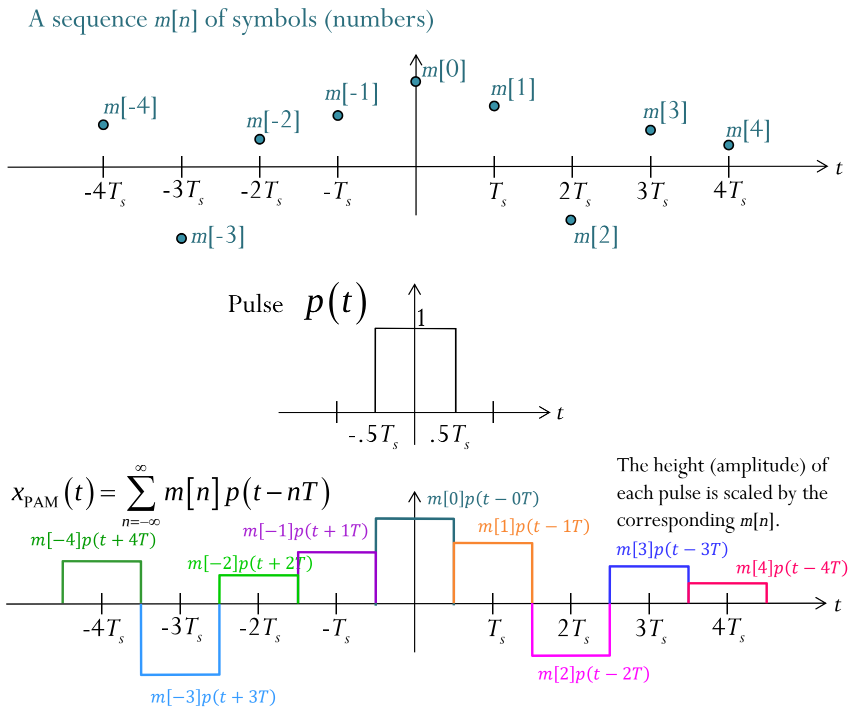


Figure 58: An example of a PAM signal

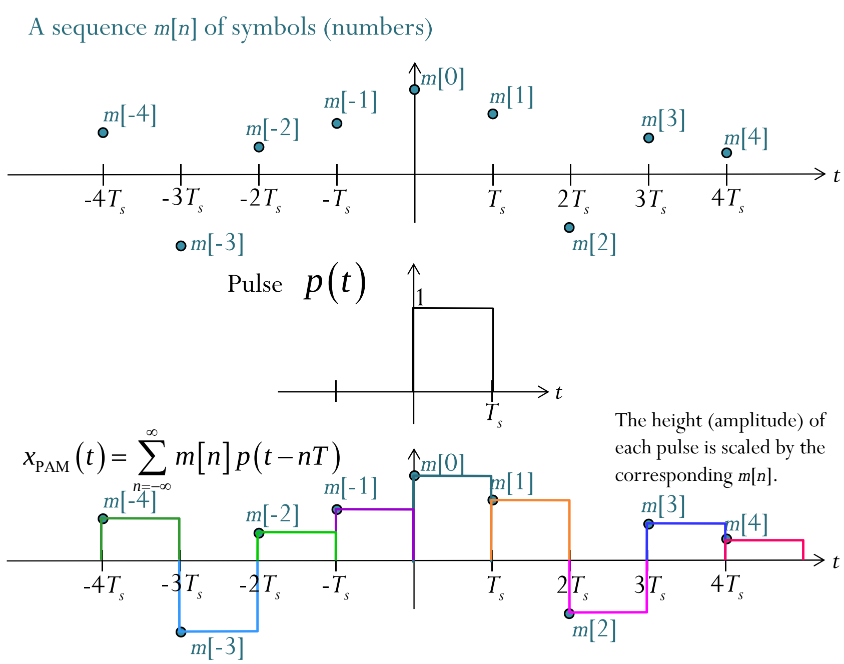


Figure 59: Another Example of a PAM signal

7.13. One advantage of using pulse modulation is that it permits the simultaneous transmission of several signals on a **time-sharing** basis.

- When a pulse-modulated signal occupies only a part of the channel time, we can transmit several pulse-modulated signals on the same channel by interleaving them.
- One User: TDM (time division multiplexing).
    - Transmit/multiplex multiple streams of information simultaneously.
- Multiple Users: TDMA (time division multiple access).

Example 7.14.

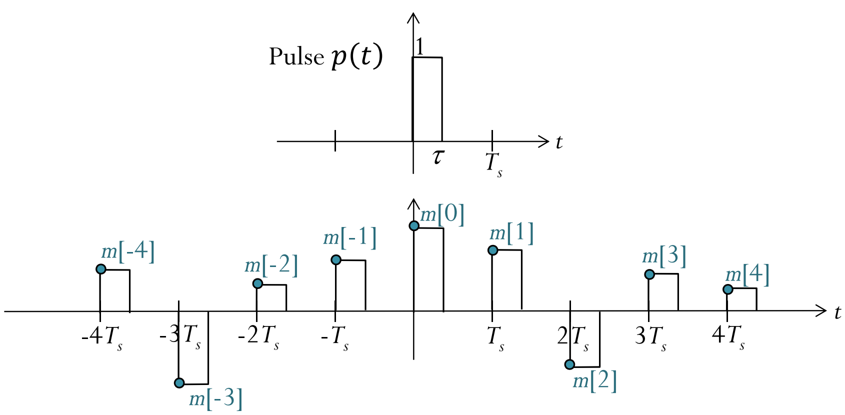


Figure 60: Time sharing in PAM

7.15. Frequency-Domain Analysis of PAM:
We start with Equation (85) in Definition 7.10.

$$x_{\mathrm{PAM}}(t) = \sum_{n=-\infty}^{\infty} m[n]\, p(t - nT_s)$$

(a) Method 1: By the time-shift property,

$$p(t - nT_s) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} P(f)\, e^{-j2\pi f n T_s}.$$

Applying the linearity property of Fourier transform to $x_{\mathrm{PAM}}(t) = \sum_n m[n]\, p(t - nT_s)$, we have

$$X_{\mathrm{PAM}}(f) = \sum_n m[n]\, P(f)\, e^{-j2\pi f n T_s} = P(f) \sum_n m[n]\, e^{-j2\pi f n T_s}.$$

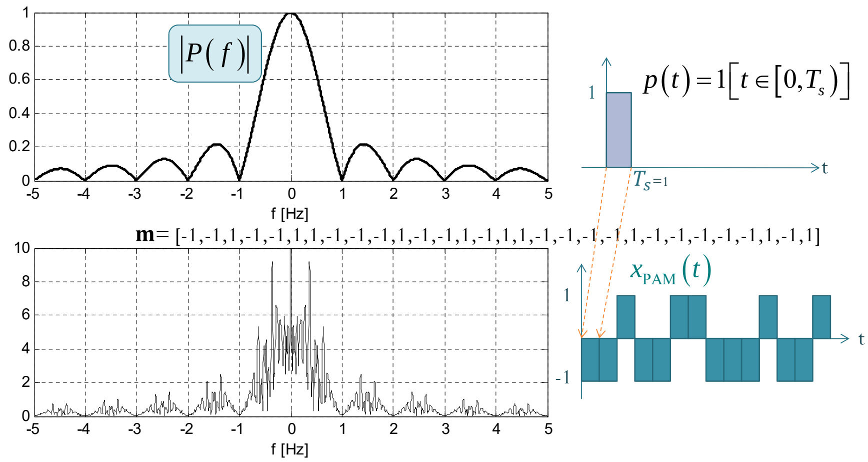


Figure 61: Frequency-Domain Analysis of PAM

The oscillation in the complex-exponential function

$$e^{-j2\pi(nT_s)f} = \cos\left(2\pi\left(nT_s\right)f\right) - j\sin\left(2\pi\left(nT_s\right)f\right)$$

adds the oscillation on top of the spectrum $P(f)$ of the pulse. The larger the value of n, the faster $X_{\text{PAM}}(f)$ oscillates in the frequency domain.

(b) Method 2: Alternatively, one may express

$$\begin{aligned} x_{\text{PAM}}(t) &= \sum_{n=-\infty}^{\infty} m[n]\, p(t - nT_s) = \sum_{n=-\infty}^{\infty} m[n]\, p(t) * \delta(t - nT_s) \\ &= p(t) * \left(\sum_{n=-\infty}^{\infty} m[n]\, \delta(t - nT_s) \right). \end{aligned}$$

Recall, from Definition 6.20, that $\sum_{n=-\infty}^{\infty} m[n]\, \delta(t - nT_s)$ can be denoted by $m_\delta(t)$ if we think of the sequence $m[n]$ as originally comes from a continuous-time signal $m(t)$. In which case

$$x_{\text{PAM}}(t) = p(t) * m_\delta(t)$$

and

$$X_{\text{PAM}}(f) = P(f)\, M_\delta(f).$$

7.3 Inter-symbol Interference and Pulse Shaping

7.16. Continue from the previous section. The generated PAM signal $x(t)$ would be transmitted via the communication channel which usually corrupts it. At the receiver, the received signal is $y(t)$. A more advanced receiver would try to first cancel the effect of the channel. However, for simplicity, let's assume that our receiver simply samples $y(t)$ every T seconds to get

$$y[n] = y(t)|_{t=nT}$$

and we will take this to be the estimate $\hat{m}[n]$ of our $m[n]$.

- In this section, we drop the subscript s from T_s.
- If $m[n]$ is the sampled version of $m(t)$, then at the receiver, after we recover $m[n]$, we can reconstruct $m(t)$ by using the reconstructing equation (84).

Because our assumed receiver is so simple, we are going to also assume[26] that $y(t) = x(t)$.

7.17. Our goal is to design a "good" pulse $p(t)$ that satisfies two important properties

(a) $\hat{m}[n] = m[n]$ for all n. Under our assumptions above, this means we want $x[n] \equiv x(nT) = m[n]$ for all n.

(b) $P(f)$ is band-limited and hence $X(f)$ is band-limited.

We will first give examples of "poor" $p(t)$.

[26] Alternatively, we may assume that there is an earlier part of the receiver that (perfectly) eliminates the effect of the channel for us.

Example 7.18. Let's consider the rectangular pulse used in Figure 62 in which

$$p(t) = 1[|t| \leq T/2].$$

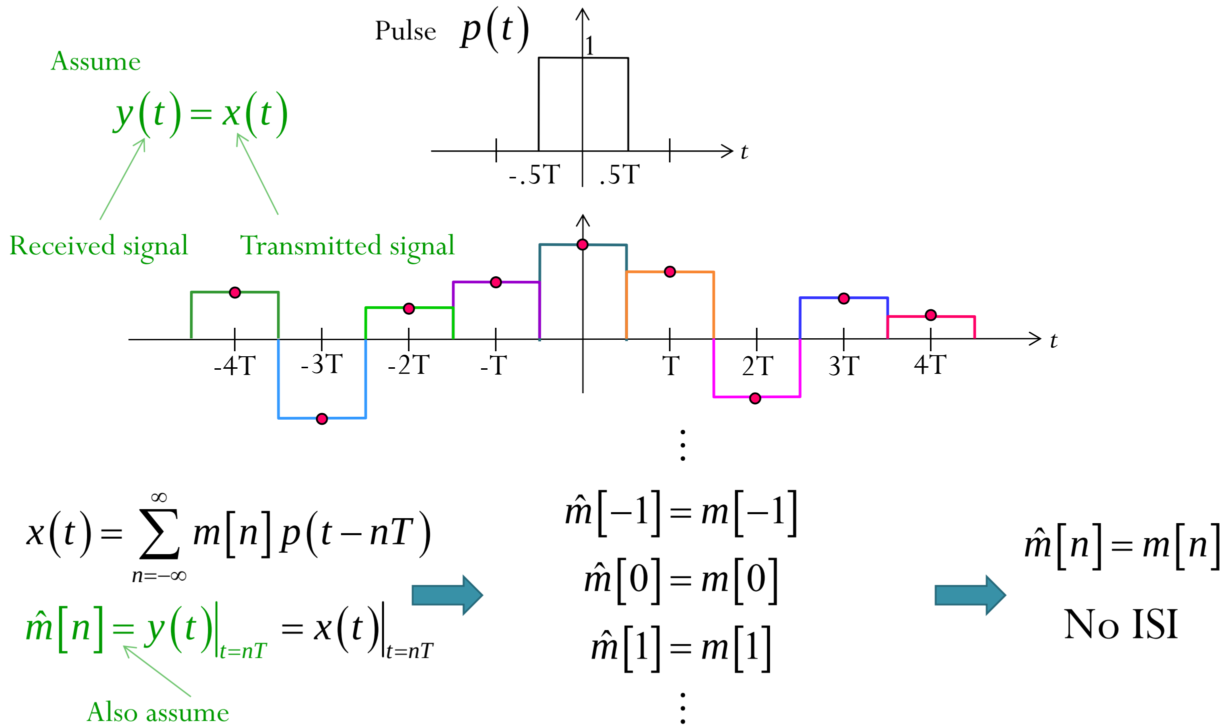


Figure 62: PAM

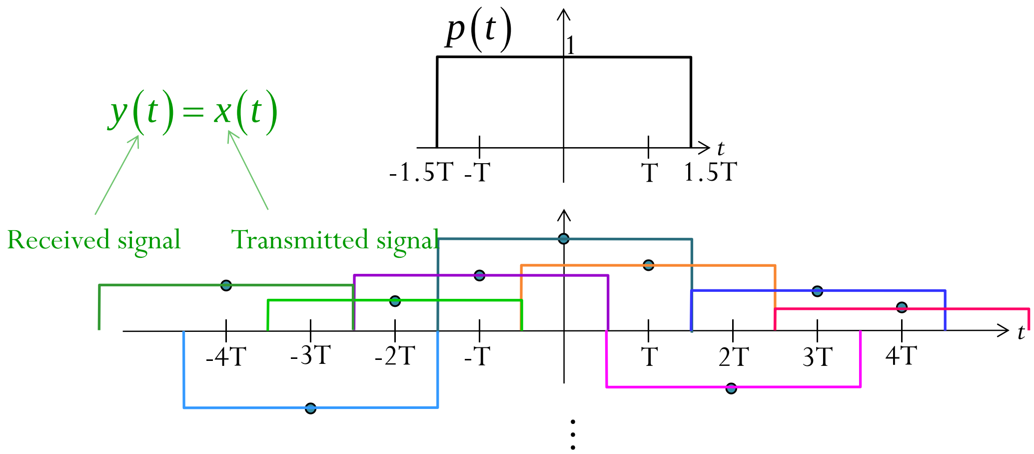
(a) $\hat{m}[n] = m[n]$ for all n.

(b) The Fourier transform of the rectangular pulse is a sinc function. So, it is not band-limited.

Example 7.19. Let's try a wider rectangular pulse:

$$p(t) = 1[|t| \leq 1.5T].$$

Figure 63 illustrates that we face a problem called **inter-symbol interference** (**ISI**) in our sequence $\hat{m}[n]$ at the receiver. The pulses are too wide; they interfere with other pulses at the sampling time instants (decision making instants), making $\hat{m}[n] \neq m[n]$.



$$x(t)=\sum_{n=-\infty}^{\infty} m[n]\,p(t-nT)$$

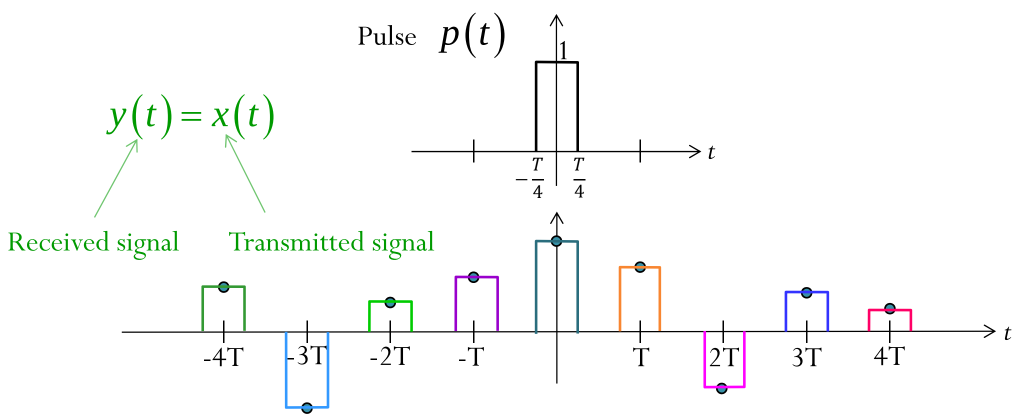
$$\hat{m}[n]=y(t)\big|_{t=nT}=x(t)\big|_{t=nT}$$

$$\hat{m}[-1]=m[-2]+m[-1]+m[0]$$
$$\hat{m}[0]=m[-1]+m[0]+m[1]$$
$$\hat{m}[1]=m[0]+m[1]+m[2]$$
$$\vdots$$

$$\hat{m}[n]\neq m[n]$$

Suffer ISI

Figure 63: ISI in PAM



$$x(t)=\sum_{n=-\infty}^{\infty} m[n]\,p(t-nT)$$

$$\hat{m}[n]=y(t)\big|_{t=nT}=x(t)\big|_{t=nT}$$

$$\hat{m}[-1]=m[-1]$$
$$\hat{m}[0]=m[0]$$
$$\hat{m}[1]=m[1]$$
$$\vdots$$

$$\hat{m}[n]=m[n]$$

No ISI

Figure 64: PAM using narrow pulses

Example 7.20. $p(t) = 1[|t| \leq T/4]$ is used in Figure 64.

- When the pulse $p(t)$ is narrower than T, we know that the pulses in PAM signal will not overlap and therefore we won't have any ISI problem.

Example 7.21. Even when the pulses are wider than T, if they do not interfere with other pulses at the sampling time instants (decision making instants), we can still have no ISI.

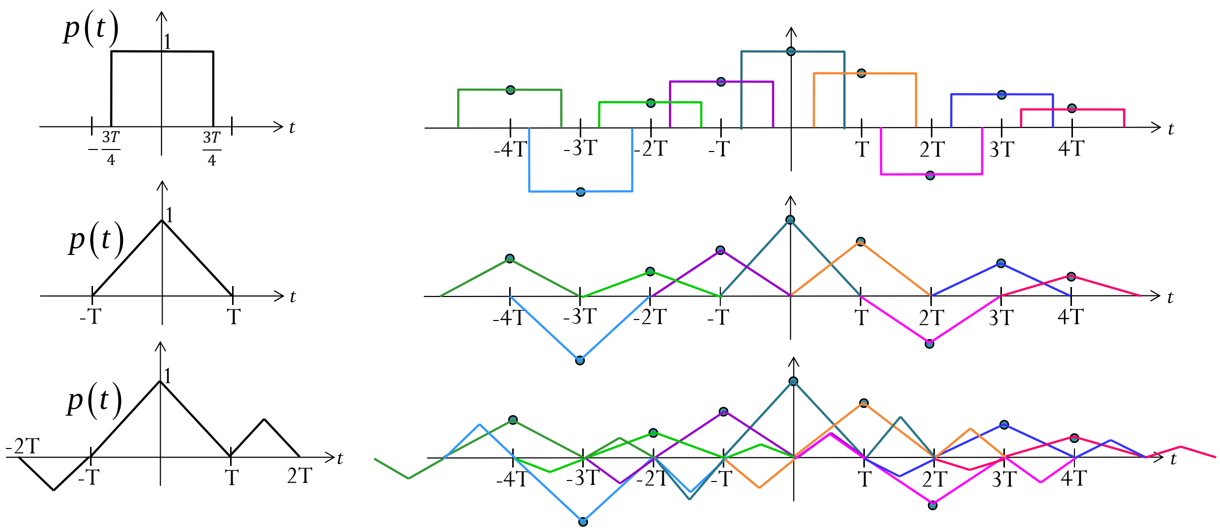


Figure 65: Examples of pulses that do not cause ISI.

7.22. We can now conclude that a "good" pulse satisfying condition (a) in 7.17 must not cause **inter-symbol interference (ISI)**: at the receiver, the nth symbol $\hat{m}[n]$ should not be affected by the preceding or succeeding transmitted symbol $m[k]$, $k \neq n$. This requirement means that a "good" pulse should have the following property:

$$p(t)|_{t=nT} = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases} \tag{86}$$

Combining this with condition (b) in 7.17, we then want "band-limited pulses specially shaped to avoid ISI (by satisfying (86))" [3, p 506].

7.23. An obvious choice for such $p(t)$ would be the sinc function that we used in the reconstruction equation (84):

Recall Figure 51, repeated here (with modified labels) as Figure 66.

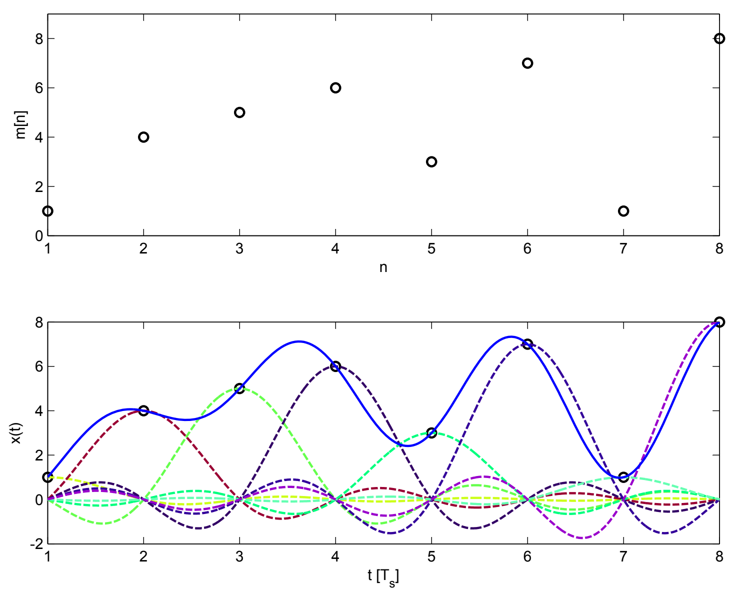


Figure 66: Using the sinc pulse in PAM

Practically, there are problems that force us to seek better pulse shape.

(a) Infinite duration

(b) Steep slope at each 0-intercept.

(c) $\max_t \{x(t)\}$ could be a lot larger than $\max_n \{m[n]\}$.

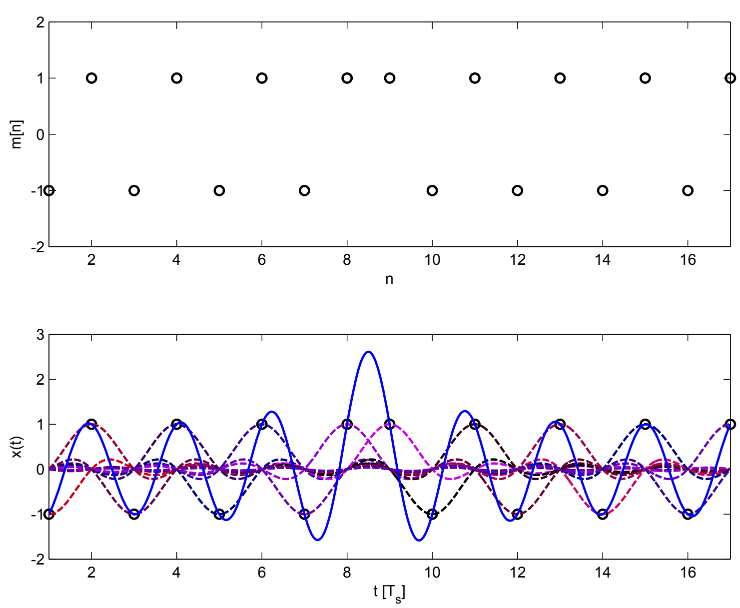


Figure 67: Using the sinc pulse in PAM can cause high peak.

7.24. Because the sinc function may not be a good choice, we now have to consider other pulses that are band-limited and also satisfy (86). To check that a signal is band-limited, we need to look in the frequency domain. However, condition (86) is specified in the time domain. Therefore, we will try to translate condition (86) into a requirement in the frequency domain.

7.25. Note that condition (86) considers $p(t)|_{t=nT}$ which can be thought of as the samples $p[n]$ of the pulse $p(t)$ where the sampling period is $T_s = T$. Recall, from (83), that

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g[n]\delta(t - nT_s) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} G_\delta(f) = f_s \sum_{k=-\infty}^{\infty} G(f - kf_s).$$

Therefore,

$$p_\delta(t) = \sum_{n=-\infty}^{\infty} p[n]\delta(t - nT) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} P_\delta(f) = \frac{1}{T}\sum_{k=-\infty}^{\infty} P\left(f - \frac{k}{T}\right). \qquad (87)$$

On the LHS, by condition (86), the only nonzero term in the sum is the one with $n = 0$. Therefore, condition (86) is equivalent to $p_\delta(t) = \delta(t)$. However, recall that $\delta(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftharpoons}} 1$. Therefore, we must have $P_\delta(f) \equiv 1$. Hence, to check condition (86), we can equivalently check that the RHS of (87) must be $\equiv 1$.

Note that $P_\delta(f)$ is "periodic" (in the freq. domain) with "period" $\frac{1}{T}$. (Recall that $G_\delta(f)$ is "periodic" (in the freq. domain) with "period" f_s.) Therefore, the checking does not need to be performed across all frequency f. We only need to focus on one period: $|f| \le \frac{1}{2T}$.

This observation is formally stated as the "Nyquist's criterion" below.

7.26. Nyquist's (first) Criterion for Zero ISI: A pulse $p(t)$ whose Fourier transform $P(f)$ satisfies the criterion

$$\sum_{k=-\infty}^{\infty} P\left(f - \frac{k}{T}\right) \equiv T, \qquad |f| \le \frac{1}{2T} \qquad (88)$$

has sample values satisfying condition (86):

$$p[n] = p(t)|_{t=nT} = \begin{cases} 1, & n = 0, \\ 0, & n \ne 0. \end{cases}$$

- Using this pulse, there will be no ISI in the sample values of $y(t)$ because

$$y[n] = y(t)|_{t=nT} = \sum_{k=-\infty}^{\infty} m[n]\, p(t-kT)\Big|_{t=nT} = \sum_{k=-\infty}^{\infty} m[n]\, p(nT-kT)$$
$$= \sum_{k=-\infty}^{\infty} m[n]\, p[n-k] = m[n]$$

Definition 7.27. A pulse $p(t)$ is a **Nyquist pulse** if its Fourier transform $P(f)$ satisfies (88) above.

Example 7.28. We know that the sinc pulse we used in Example 7.23 works (causing no ISI). Let's check it with the Nyquist's criterion:

Example 7.29.

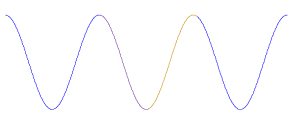
Example 7.30.

Example 7.31.

Example 7.32. An important family of Nyquist pulses is called the **raised cosine** family. Its Fourier transform is given by

$$P_{\mathrm{RC}}(f;\alpha) = \begin{cases} T, & 0 \le |f| \le \frac{1-\alpha}{2T} \\ \frac{T}{2}\left(1+\cos\left(\frac{\pi T}{\alpha}\left(|f| - \frac{1-\alpha}{2T}\right)\right)\right), & \frac{1-\alpha}{2T} \le |f| \le \frac{1+\alpha}{2T} \\ 0, & |f| \ge \frac{1+\alpha}{2T} \end{cases}$$

with a parameter α called the **roll-off factor**.



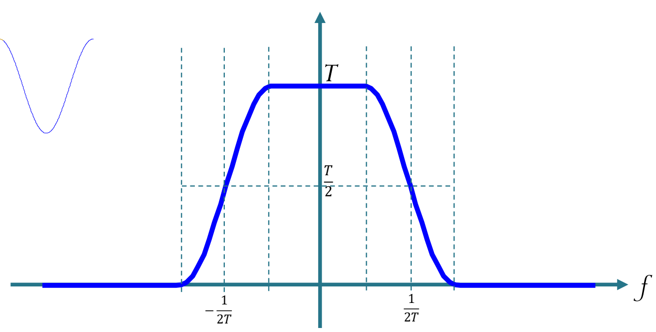


Figure 68: Raised cosine pulse (in the frequency domain)

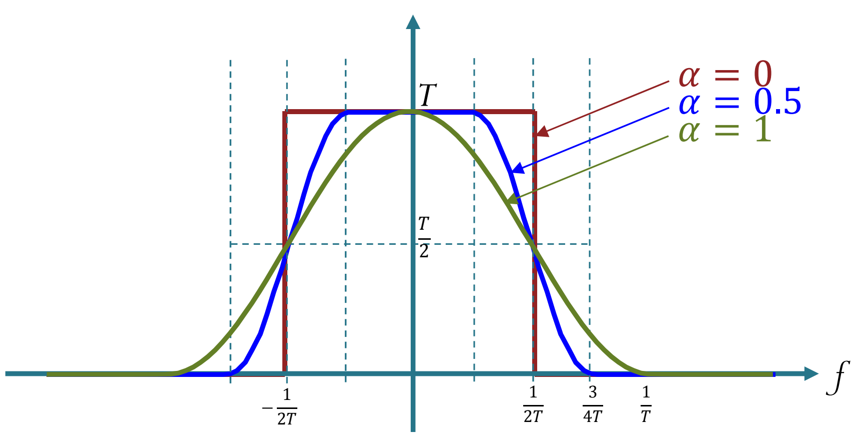


Figure 69: Raised cosine pulse (in the frequency domain) with different values of the roll-off factor

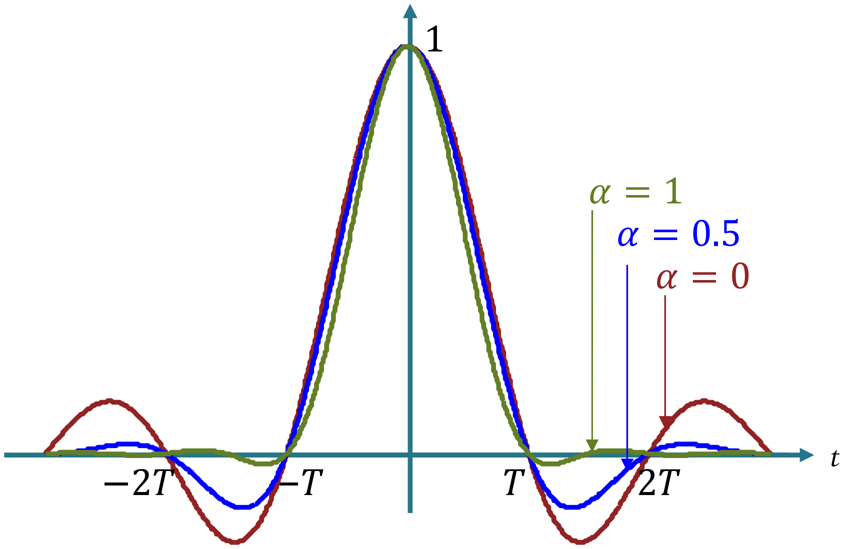


Figure 70: Raised cosine pulse (in the time domain) with different values of the roll-off factor

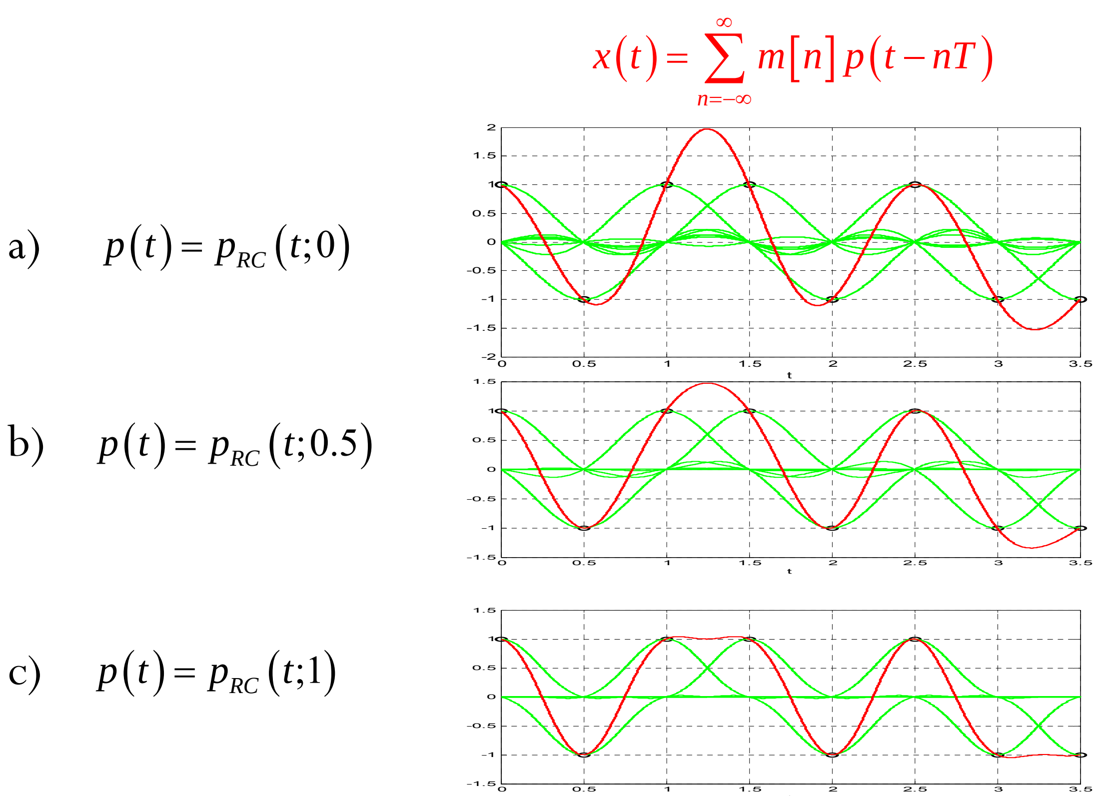


Figure 71: Using the raised cosine pulses in PAM

7.33. A maximum of $2B$ independent pieces (samples or symbols) of information per second can be transmitted, errorfree, over a noiseless channel of bandwidth B Hz [5, p 260].

- Symbol rate of $R_s = 2B$ is achievable:
  - Start with $2B$ pieces of information per second. Denote the sequence of such information by m_n.
  - Construct a signal $m(t)$ whose (Nyquist) sample values $m[n] = m\left(n\frac{1}{2B}\right)$ agrees with m_n by the reconstruction equation (84).
    * The reconstruction equation uses linear combination of the (scaled and time-shifted) sinc function that are all band-limited to B. So, $m(t)$ will also be band-limited to B.
- Symbol rate of $R_s > 2B$ is not achievable.
  - Note that a signal that is bandlimited to $B < 0.5R_s$ cannot be a Nyquist pulse.
  - If $R_s > 2B$, we have $0.5R_s > B$.

7.34. A bandpass signal whose spectrum exists over a frequency band $f_c - \frac{B}{2} < |f| < f_c + \frac{B}{2}$ has a bandwidth B Hz. Such a signal is also uniquely determined by samples taken at above the Nyquist frequency $2B$. The sampling theorem is generally more complex in such case. It uses two interlaced sampling trains, each at a rate of $f_s > B$ samples per second (known as second-order sampling). [6, p 304]